3C36: COSMOLOGY and EXTRAGALACTIC ASTRONOMY

Problem Paper 1 – Notes for Answers

[marks]

Question 1

(*a*) *Suppose, for the purposes of a rough calculation, that the Milky Way galaxy contains 10^{11} stars (and nothing else), each weighing one solar mass on average, and each composed entirely of hydrogen (not unreasonable for an order-of-magnitude estimate). Suppose further that the Milky Way is a typical galaxy, and that each cubic megaparsec of space contains 1 galaxy. On average, what volume is occupied by a single hydrogen atom?*

The number of hydrogen atoms per cubic Mpc is (total mass)/(hydrogen-atom mass),

$$\begin{aligned} N(\mathrm{H}) &= 10^{11} \times M_{\odot}/m_{\mathrm{H}} \\ &= 10^{11} \times 1.99 \times 10^{30}\ \mathrm{kg}/1.66 \times 10^{-27}\ \mathrm{kg} = 1.199 \times 10^{68} \end{aligned}$$

and the volume occupied by one atom is $(1/N(\mathrm{H}))$ Mpc^{-3} – i.e., 8.34×10^{-69} Mpc^{-3}, or, in more sensible units,

$$8.34 \times 10^{-69} \times (3.086 \times 10^{22})^3\ \mathrm{m}^{-3} = 0.25\ \mathrm{m}^{-3}$$

(since 1 Mpc $= 3.086 \times 10^{22}$ m). [3]

[Very roughly speaking, on average there's of order one atom per cubic metre in the local universe (as compared with roughly one atom per cubic cm in interstellar space in the Galaxy).]

(*b*) *If the age of the Universe is* 1.4×10^{10} *years, estimate the volume of the observable universe. (Remember, this is intended to be a* rough *calculation, so the only extra information you should need to make this estimate is the speed of light.)*

If the age of the universe is t_0, we can see to a distance of something like $c \times t_0$:

$$1.4 \times 10^{10} \times (365.25 \times 24 \times 60 \times 60) \times 3.00 \times 10^{8} = 1.325 \times 10^{26}\ \mathrm{m};$$

that is, a volume of 9.75×10^{78} m^3, or, in more sensible units, 3.32×10^{11} Mpc^3. [2]

[The actual size of the observable universe is not simply ct_0, but depends on the detailed cosmological model. In general, the radius of the observable universe is somewhat larger than the simple estimate (by about a factor of 2), because the universe expanded while the light traveled across is. An alternative way of looking at this is to note that we see distant objects where they were, not where they are (and they're now further away).]

(*c*) *The Standard EuroBeach is 1km long, 10m across, and 1m deep; and the European Standard Sandgrain occupies 1 cubic mm. Which is the larger number: the number of galaxies in the observable universe, or the number of grains of sand on a beach? What about the number of stars compared to the number of sandgrains?*

The volume of the Eurobeach is $1000 \times 10 \times 1 = 10^4$ m^3;
the volume occupied by the Standard Sandgrain is 1 $\mathrm{mm}^3 = 10^{-9}$ m3;
whence the number of grains on the beach is $10^4/10^{-9} = 10^{13}$.

Taking the volume of the observable universe as 3.32×10^{11} Mpc^3 (from above), then if there is one galaxy Mpc^{-3}, and 10^{11} stars in a galaxy (as stated in part (*a*)), then there are 3.32×10^{22} stars in the observable Universe.

So – very roughly speaking:

- the number of stars in a galaxy is about the same as the number of galaxies in the universe;
- there are more sandgrains on a beach than stars in a galaxy (or galaxies in the universe);
- but there are many, many fewer sandgrains on a beach – or, for that matter, on all the beaches of the world combined – than there are stars in the observable universe. [3]

[You'll see that this is a very simple – indeed, almost trivial – calculation; and yet not so long ago it was "announced" in a press release:

http://news.bbc.co.uk/1/hi/sci/tech/3085885.stm

Try this Google search:

http://www.google.co.uk/search?q=simon+driver+stars+in+the+universe

to see the astonishing amount of media coverage this "discovery" attracted!]

Question 2
As discussed in lectures, X-ray emitting intergalactic gas in clusters of galaxies can be used to estimate $M(r)$, the total cluster mass within some radius r. This involves combining the equation of hydrostatic equilibrium for the gas,

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{GM(r)\rho(r)}{r^2} \tag{1}$$

with an equation of state,

$$p(r) = \frac{\rho(r)kT(r)}{m}, \tag{2}$$

to obtain

$$M(r) = \frac{-kT(r)r}{Gm}\left\{\frac{\mathrm{d}\ln\rho(r)}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln T(r)}{\mathrm{d}\ln r}\right\} \tag{3}$$

(where $p(r)$, $\rho(r)$, and $T(r)$ are the pressure, density, and temperature at radius r, m is the mean particle mass, and k is Boltzmann's constant).
By differentiating equation (2), and using equation (1), derive equation (3).

From (1) and (2),

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\rho(r)kT(r)}{m}\right] &= \frac{GM(r)\rho(r)}{r^2} \\
\frac{kT(r)}{m}\frac{\mathrm{d}\rho(r)}{\mathrm{d}r} + \frac{k\rho(r)}{m}\frac{\mathrm{d}T(r)}{\mathrm{d}r} &= \frac{GM(r)\rho(r)}{r^2}
\end{aligned}$$

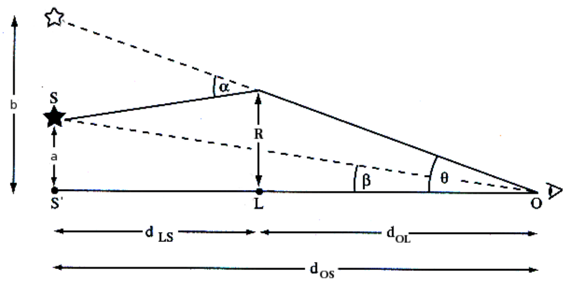
Rearranging, and reusing eqtn.2,

$$\frac{kT(r)r^2}{Gm\rho(r)}\frac{\mathrm{d}\rho(r)}{\mathrm{d}r} + \frac{k}{Gm}\frac{T(r)}{T(r)}r^2\frac{\mathrm{d}T(r)}{\mathrm{d}r} = -M(r);$$

that is,

$$\begin{aligned}
M(r) &= \frac{-kT(r)r}{Gm}\left\{\frac{r}{\rho(r)}\frac{\mathrm{d}\rho(\mathrm{r})}{\mathrm{dr}} + \frac{r}{T(r)}\frac{\mathrm{dT(r)}}{\mathrm{dr}}\right\} \\
&= \frac{-kT(r)r}{Gm}\left\{\frac{\mathrm{d}\ln\rho(r)}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln T(r)}{\mathrm{d}\ln r}\right\}
\end{aligned}$$

QED [5]



Question 3

By expressing the lengths a and b in terms of the angles β and θ, show that the observed angular displacement in this generalized case is given by...

The angles in this geometry are small (i.e., $\sin(x) \simeq \tan(x) \simeq x$, when x is measured in radians), so

$$\begin{aligned} \beta &\simeq \tfrac{a}{d_{\mathrm{OS}}} & & \\ \theta &\simeq \tfrac{b}{d_{\mathrm{OS}}} &= \frac{R}{d_{\mathrm{OL}}} & \\ \alpha &\simeq \tfrac{(b-a)}{d_{\mathrm{LS}}} &\equiv \frac{4GM}{c^2 R} &= \frac{4GM}{c^2}\frac{1}{\theta d_{\mathrm{OL}}} \end{aligned} \tag{4}$$

whence

$$\begin{aligned} \theta - \beta &= \frac{b-a}{d_{\mathrm{OS}}} = \frac{\alpha d_{\mathrm{LS}}}{d_{\mathrm{OS}}} = \frac{4GM}{c^2}\frac{1}{\theta d_{\mathrm{OL}}}\frac{d_{\mathrm{LS}}}{d_{\mathrm{OS}}} \\ &= \frac{1}{\theta}\left[\frac{4GM}{c^2}\frac{d_{\mathrm{LS}}}{d_{\mathrm{OL}} d_{\mathrm{OS}}}\right] \end{aligned}$$

QED [5]

(*b*) *Use the above equation to obtain an estimate the angular deflection of a distant star whose position, in the absence of gravitational lensing, is exactly on the limb of the Sun. Assume that the Sun's distance is 1 Astronomical Unit, and that the angular deflection is small compared to the Sun's angular radius of 15′ (which, for the purposes of this question, can be treated as a 'small angle').*

For a "distant" star, $d_{\mathrm{LS}} >> d_{\mathrm{OL}}$ and so

$$\frac{d_{\mathrm{LS}}}{d_{\mathrm{OL}} d_{\mathrm{OS}}} \simeq \frac{d_{\mathrm{OS}}}{d_{\mathrm{OL}} d_{\mathrm{OS}}} = \frac{1}{d_{\mathrm{OL}}}.$$

We don't know β, only θ (the "position, in the absence of gravitational lensing, is exactly on the limb of the Sun"), but since "the angular deflection is small compared to the Sun's angular radius", $1/\theta \simeq 1/\beta$. Our equation for the deflection is therefore

$$\theta - \beta \simeq \frac{1}{\beta}\left[\frac{4GM_\odot}{c^2}\frac{1}{d_{\mathrm{OL}}}\right]$$

with $d_{\mathrm{OL}} = 1$ AU and $\beta = 15' = (15/60) \times (\pi/180)$ radians:

$$\begin{aligned} \theta - \beta &\simeq \frac{60 \times 180}{15\pi}\left[\frac{4 \times (6.67 \times 10^{-11}) \times (1.989 \times 10^{30})}{(2.9979 \times 10^8)^2}\frac{1}{1.496 \times 10^{11}}\right] \qquad \frac{\mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2}\quad \mathrm{kg}}{(\mathrm{m\ s}^{(-1)})^2\quad \mathrm{m}} \\ &= 9.0455 \times 10^{-6} \text{ radians} = 1.87'' \end{aligned}$$

[5]